Homework 3 (Due 2/12/2014)

Math 622

February 7, 2014

Fixed minor typos in problems 4, 5. A hint given in problem 6.

Problem 1 is an interesting fact related to Corollary 11.5.3, which you should definitely read. You are not required to hand in problem 1.

Problem 2 is meant to help you understand the calculations involved in Theorem 11.7.5, and go over methods you will need for later problems.

It is useful for some of the calculations to know that if Y is a normal random variable with mean μ and variance σ^2, its moment generating function is $E[e^{uY}] = e^{\mu u+\sigma^2 u^2/2}$. Note also that if Z is a Poisson random variable with parameter ν, $E[Z] = \nu$ and $Var(Z) = \nu$.

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1. Read Corollary 11.5.3. The following problem is to prove a similar result for two Poisson processes rather than a Poisson process and a Brownian motion. Let N_1 and N_2 be Poisson processes relative to the same filtration. Let their rates be λ_1 and λ_2. Assume they have no simultaneous jumps. Show they are independent, as follows. Show that the calculation of problem 2 of Assignment 2 is valid and deduce from it that the process defined there is a martingale for each u_1 and u_2. (Recall Exercise 3 from Assignment 2.)

2. *Some probability theory and an application to compound Poisson processes.* Let Z be a random variable. Then by the averaging property of conditional expectation, $E[U] = E\big[E[U\big|Z]\big]$. (For example, see equation (2.3.17) in Shreve with $A = \Omega$ and $\mathcal{G} = \sigma(Z)$.)

Let $Q(t) = \sum_{j=1}^{N(t)} Y_i$ be a compound Poisson process, so that N is a Poisson process with rate λ and $Y_1, Y_2, \ldots$ is a sequence of independent, identically distributed random variables that are independent of N.

a) Consider evaluating $E[G(Q(t))]$ for some function G. For every integer $n \geq 0$, let $\ell(n) := E\Big[G\big(\sum_{j=1}^{n} Y_i\big)\Big]$ (when $n = 0$, interpret this as $\ell(0) = G(0)$). By conditioning on $N(t)$—that is, let $N(t)$ play the role of Z above—show that

$$E[G(Q(t))] = \sum_{n=0}^{\infty} \ell(n) \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

b) Assume that $Y_1, Y_2 \ldots$ are independent, normal random variables with mean μ and variance σ^2. Explain why the condition distribution of $Q(t)$ given $N(t) = n$ is normal with mean $n\mu$ and variance $n\sigma^2$. Using the technique of part a), show that,

$$E[Q^2(t)] = (\sigma^2 + \mu^2)\lambda t + (\mu \lambda t)^2,$$

and that,

$$E[e^{uQ(t)}] = e^{\lambda t(e^{u\mu + \sigma^2 u^2/2} - 1)}.$$

c) Prove as a general principle. Assume that Y is independent of $X_1, \ldots, X_M$. Show that $E\big[H(Y, X_1, \ldots, X_M)\big] = E\big[h(X_1, \ldots, X_M)\big]$, where $h(x_1, \ldots, x_M) = E\big[H(Y, x_1, \ldots, x_M)\big]$.

3. *Merton's jump diffusion process.* Let $Z_1, Z_2, \ldots$ be independent standard normal random variables that are independent of W and N, where W and N are independent, W is a Brownian motion, and N is a Poisson process with rate λ. Let $Q(t) = \sum_1^{N(t)} [e^{Z_i} - 1]$. Consider the price model

$$dS(t) = \alpha S(t)\, dt + S(t)\sigma\, dW(t) + S(t-) dQ(t),$$

This price process is called Merton's jump diffusion.

a) Explicitly identify a constant θ and a compound Poisson process $\bar{Q}$ such that

$$S(t) = S(0) \exp\{\sigma W(t) + \theta t + \bar{Q}(t)\}.$$

b) Let r be the risk-free interest rate. What must α be so that this model is risk-neutral?

c) Let α be chosen as in b), so that the model is risk-neutral. If we were to use this risk-neutral model to price a call option at strike K and expiry T, we would find the price is $V(t) = e^{-r(T-t)} E\Big[\big(S(T) - K\big)^+ \big| \mathcal{F}(t)\Big]$. Show that $V(t) = c(t, S(t))$, where $c(t, x)$ is given in the form

$$c(t, x) = e^{-r(T-t)} E\Big[H(x, Y(T-t))\Big],$$

where $Y(s)$ has the form $Y(s) = \sigma W(s) + \theta s + \bar{Q}(s)$. Your answer should explicitly define H and should express θ in terms of r, λ, and σ.

d) Let

$$\bar{\kappa}(\tau, x, \nu) := e^{-r\tau} E\Big[\big(xe^{\nu U} - K\big)^+\Big],$$

where U is a standard normal random variable. By conditioning on $N(T-t)$ in the style of Exercise 2, find constants a and $\nu_1, \nu_2, \ldots$ so that

$$c(t,x) = \sum_{n=0}^{\infty} \bar{\kappa}(T-t, ax, \nu_n) \frac{(\lambda(T-t))^n}{n!} e^{-\lambda(T-t)}.$$

(The constants a and $\nu_1, \nu_2, \ldots$ will depend on $T-t$ and other parameters of the model.)

Remark. This is an interesting formula because $\bar{\kappa}$ has the following explicit form, closely related to the Black-Scholes formula:

$$\bar{\kappa}(\tau, x, \nu) = e^{-r\tau}\Big[xe^{\nu^2/2} N\big(\frac{\nu^2 - \ln(K/x)}{\nu}\big) - KN\big(-\nu^{-1}\ln(K/x)\big)\Big].$$

4. Let $Q(t)$ be the compound Poisson process

$$Q(t) = \sum_{k=1}^{N(t)} Y_i,$$

where $Y_1, Y_2, \ldots$ are i.i.d. with $\mathbf{P}\Big(Y_i = \frac{3}{4}\Big) = \frac{3}{5}$ and $\mathbf{P}\Big(Y_i = -\frac{3}{4}\Big) = \frac{2}{5}$, and where N is a Poisson process with rate 2. Let $N_1(t)$ count the number of jumps of Q by $3/4$ and let $N_2(t)$ count the number of jumps of Q by $-3/4$.

Consider,

$$dS(t) = -(3/10)S(t)\,dt + S(t-)\,dQ(t), \qquad S(0) = 1. \tag{1}$$

(a) If S solves equation (1), is it a martingale or not? Explain briefly.

(b) There is a function $c(t,x)$ such that $E\Big[(K - S(T))^+ \big| \mathcal{F}(t)\Big] = c(t, S(t))$. Find an explicit expression for $c(t,x)$ as a doubly infinite sum.

5. Consider the risk neutral model,

$$dS(t) = rS(t)\,dt + S(t)\,d\tilde{W}(t) + \sqrt{S(t-)}S(t-)\,d[Q(t) - t/2]$$

Here $Q(t) = N_1(t) - (1/2)N_2(t)$ is a compound Poisson process , where both N_1 and N_2 are independent Poisson processes, both with rate $\lambda = 1$. As usual, N_1 and N_2 are independent of W. In this model the jumps the price experiences in its returns are affected by the level of the price. There is no longer an explicit solution since the factor $\sqrt{S(t-)}$ appears in the last term.

This price equation has a solution which is a Markov process, and so it will be possible to write the price of a call option in the form

$$V(t) = e^{-r(T-t)}\tilde{E}\Big[(S(T) - K)^+\big|\mathcal{F}(t)\Big] = c(t, S(t))$$

The purpose of this problem is to derive an equation for c of the type found in Theorem 11.7.7 for the model (11.7.27).

a) Show that

$$S(t) = \begin{cases} S(t-), & \text{if } \triangle Q(t) = 0; \\ S(t-)\Big(1 + \sqrt{S(t-)}\Big), & \text{if } \triangle N_1(t) = 1; \\ S(t-)\Big(1 - (1/2)\sqrt{S(t-)}\Big), & \text{if } \triangle N_2(t) = 1. \end{cases}$$

and thus that

$$\begin{aligned} \triangle c(t, S(t)) &= \Big[c\big(t, S(t-)(1 + \sqrt{S(t-)}\big) - c(t, S(t-))\Big]\triangle N_1(t) \\ &\quad + \Big[c\big(t, S(t-)(1 - \sqrt{S(t-)}/2)\big) - c(t, S(t-))\Big]\triangle N_2(t) \end{aligned}$$

b) By applying Itô's rule for jump processes to $e^{-rt}c(t, S(t))$ and insisting that it be a martingale, find an equation that, approximately, is of the type derived in Theorem 11.7.7. It will look very similar, except that the terms $c(t, x(1 + y_m))$ in equation (11.7.33) will be modified.

6. Create a price model for a single asset price with the following properties:

(i) Normally the price follows a Black-Scholes type of evolution with a constant volatility.

(ii) Occasionally the price jumps up by an amount that is, on average, one quarter of the pre-jump price. These jumps arrive according to a Poisson process.

(iii) Occasionally the price jumps down by an amount that is, on average, one third of the pre-jump price. These price jolts arrive according to a Poisson process independently of the positive jumps.

(iv) The positive jumps arrive at a faster rate than the negative jumps.

There is no one right answer to this problem. Your model will contain different parameters. Try to leave as many as possible as free constants—you would want to be able to choose these parameters to fit empirical data if you were to implement the model. However, the conditions (i)—(iv) might imply relation(s) among the parameters and you should specify these. *You may want to utilize this fact in the construction of the model: the sum of 2 compound Poisson processes is still a coumpound Poisson process in the following sense: if* $N_1(t), N_2(t)$ *are indendent Poisson processeswith rates* λ_1, λ_2, X_i *i.i.d and* Y_i *i.i.d random variables such that* $\{X_i\}$ *is independent of* $\{Y_i\}$ *then*

$$Q(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{j=1}^{N_2(t)} Y_j$$

is also a compound Poisson process. Can you see why? Can you write $Q(t)$ *in the regular form of a Compound Poisson process?*